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On the Focusing of Gravitational Radiation

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Abstract

We investigate the gain in intensity that can be achieved by using a massive object as a 'lens' to focus gravitational radiation incident on the object from a point-like source. An object of mass M produces a gain in intensity of the order of $\alpha GM/\lambda c^2$ where α is a numerical factor which depends on the mass distribution and λ is the wavelength of the radiation. For large mass, the gain is large, but occurs only in a beam of small angular width.

1. *Introduction*

Some experimental evidence that gravitational radiation is incident on the earth, possibly coming from the galactic center, has been reported by Weber (1970). Under the assumption that, on the average, the radiation is emitted isotropically from near the galactic center, the gravitational energy radiated has been estimated as $10^3 M_\odot$ per year; this is in disagreement with a limit of $200M_\odot$ per year set by astronomical observations (Sciama *et al.*, 1969). Furthermore, each burst of gravitational radiation, if isotropic, carries an energy of several solar masses which is astonishingly large for a single event. One obvious solution to this embarrassing problem is to suppose that the radiation is not isotropic, but somehow concentrated in the direction of the earth. Calculations of the focusing of gravitational radiation by the gravitational fields surrounding a massive object have been given by Lawrence (1971, 1973) and by Campbell & Matzner (1973). These calculations were based on an analogy with the focusing of light by gravitational fields (Liebes, 1964). A weak gravitational wave of short wavelength propagates along null geodesics and is deflected in the same way as light. The deflection of the radiation in the gravitational field of a star results in an increase of the observed intensity if the star is on, or near, the straight line between the source of radiation and the observer. The gravitational field acts as a 'lens'; however, since for rays passing through the exterior gravitational fields of a star, the deflection angle *decreases* with impact parameter, the 'lens' does not form a true focal point but only a caustic line.

The present paper investigates two effects that may be relevant in the case of focusing of gravitational radiation, but are not in the case of focusing of light. First, since the wavelength of gravitational radiation is much larger than that of light ($\lambda \simeq 2 \times 10^7$ cm for Weber's detector), it is possible for some sources (e.g., neutron stars) to behave as *point* sources; the maximum focused intensity is then limited by diffraction effects rather than by the surface brightness of the source. Second, gravitational radiation can pass through the interior of a star; under these conditions the deflection angle can *increase* with impact parameter and a true focal point is formed.

The following calculations assume that the gravitational field of the 'lens' is weak and that the deflection angle is small. It is also assumed that the gravitational wave is weak and has a wavelength short compared to the size of the 'lens'. How the radiation is produced will not concern us; the only properties of the source that are relevant to our calculation are that the source be small compared to a wavelength and that it illuminate the 'lens' fairly uniformly.

2. *Deflection of a Ray*

In a gravitational field, weak gravitational waves of sufficiently short wavelength move along null geodesics (Isaacson, 1968). This means that the wave vector k^μ, which is a null vector, undergoes parallel transport,

$$dk^\mu = -\Gamma^\mu{}_{\alpha\beta} k^\alpha \, dx^\beta \tag{2.1}$$

We assume that the gravitational field has cylindrical symmetry about the z-axis. The deflection of a ray incident from $z = -\infty$ with an impact parameter b can be calculated by integrating (2.1) from $z = -\infty$ to $z = +\infty$. For the case of small deflection (weak gravitational field at impact parameter b), the calculation may be carried out by methods familiar from Bohr's theory of the energy loss in collisions between fast charged particles. The deflection angle is given by the simple formula

$$\theta(b) = \frac{4G}{bc^2} \times (\text{mass inside impact parameter } b) \tag{2.2}$$

If b is larger than the radius of the star (of mass M), equation (2.2) becomes

$$\theta(b) = \frac{4GM}{bc^2} \tag{2.3}$$

which is the familiar deflection formula for light.

For a star of uniform density with radius R the deflection of a ray with impact parameter $b < R$ is then given by (see also Lawrence, 1971a)

$$\theta(b) = \frac{4GM}{bc^2} \left[1 - \frac{(R^2 - b^2)^{3/2}}{R^3} \right] \tag{2.4}$$

where the factor in brackets represents the fraction of the volume of the star inside impact parameter b. The deflection is plotted as a function of b in Fig. 1. Since the deflection angle near $b = 0$ increases *linearly* with b, such a mass distribution will make a fairly good lens for gravitational radiation. All rays originating from some source point and entering the 'lens' not too far from the center, will come together at an image point. We will investigate the intensity in a later section.

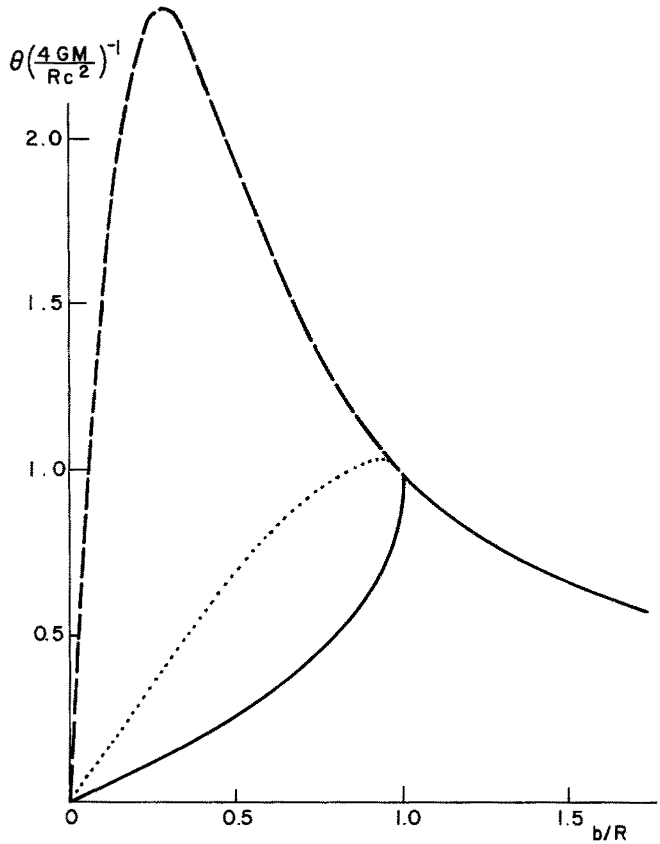

Figure 1.–Deflection angle as a function of impact parameter. Dashed line: polytrope, $n = 3$. Dotted line: sphere of uniform density. Solid line: thin shell.

For a star with a spherically symmetric mass density $\rho(r)$,

$$\theta(b) = \frac{4GM}{bc^2}\left[1 - \frac{4\pi}{M}\int_b^R r(r^2 - b^2)^{1/2}\rho\, dr\right] \tag{2.5}$$

Figure 1 shows the result of numerical evaluation of equation (2.5) for the special case of the density function

$$\rho(r) = \rho(o)[\theta_3(r)]^3 \tag{2.6}$$

where $\theta_n(r)$ is a Lane-Emden function. The density (2.6) belongs to a polytrope of index $n = 3$; this is a gas sphere held in equilibrium by radiation pressure (Chiu, 1968). As we will see later, such supermassive stars are capable of producing large increases in the intensity of gravitational waves.

Finally, we remark that the above deflection formulae are approximations for a ray that begins at $z = -l_1$ rather than at $z = -\infty$. The approximation is good provided $b^2/l_1^2 \ll 1$ and $R^2/l_1^2 \ll 1$.

3. *Intensity According to Geometrical Optics*

The intensity of a gravitational wave usually decreases according to the $1/r^2$ law as the distance from the source increases. However, if a star acting as a gravitational lens lies between the source and the observer, then the observed intensity can be much larger than the value given by the $1/r^2$ law. We first investigate the gain in the intensity by means of geometrical optics. Only modest gains are to be expected in those regions where geometrical optics is applicable; caustic surfaces and lines on which the intensity gain is very large must be investigated by wave optics.

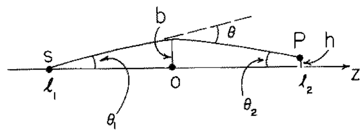

Figure 2.—Deflection of a ray.

Figure 2 shows the source S, the observer P, and the region O in which deflection occurs; all angles are greatly exaggerated. A ray arrives at $z = l_2$ at a distance h from the axis

$$h = b - l_2\theta_2 = b - l_2(\theta - \theta_1) \tag{3.1}$$

and

$$dh = l_2(1/l_1 + 1/l_2 - \theta')\,db \tag{3.2}$$

where $\theta' \equiv d\theta/db$. All the energy incident on the area $2\pi b\,db$ at the 'lens' arrives in the area $2\pi h\,dh$ at $z = l_2$. Hence the intensity at $z = l_2$ is

$$I(P) = I(O)\frac{2\pi b\,db}{2\pi h\,dh}$$

$$= I(O)\frac{b}{hl_2}\frac{1}{(1/l_1 + 1/l_2 - \theta')} \tag{3.3}$$

We define the intensity gain as $I(P)(l_1 + l_2)^2/I(O)l_1^2$. This is the factor by which the intensity is increased over that given by the $1/r^2$ law.

$$(\text{gain}) = \left(\frac{l_1 + l_2}{l_1}\right)^2 \frac{b}{hl_2} \frac{1}{(1/l_1 + 1/l_2 - \theta')} \tag{3.4}$$

For an observer on the earth, the relevant case is that with $l_2 \gg l_1$. In this case

$$(\text{gain}) \simeq \frac{\theta}{w} \frac{1}{(1 - l_1\theta')} \tag{3.5}$$

where $w = h/l_2$ is the angular distance of the observer from the axis as seen from the star.

Equation (3.4) gives a divergent result if

$$1/l_1 + 1/l_2 - \theta' = 0 \tag{3.6}$$

and also if

$$h = 0 \tag{3.7}$$

The condition (3.6) defines a surface of revolution $h = h(l_2)$, which is the focal surface or caustic surface. Condition (3.7) shows that the axis is a caustic line. In either case geometrical optics fails and it will be necessary to use wave optics.

In those regions in which geometrical optics is applicable, the intensity gain is of the order of θ/w, i.e., the intensity falls off linearly with increasing distance from the axis. Since, in the context of our weak field approximation, we must assume $\theta \ll 1$, it is clear that equation (3.5) gives an appreciable intensity increase only for small displacements from the axis.

Finally, it should be noted that there are *two* rays that reach P since it is also possible for radiation to reach P by passing below O and crossing the axis somewhere between O and P. Depending on the phase between these two rays, the intensity could be up to four times that given by (3.3), or down to zero. On the average (3.3) should be multiplied by a factor of two.

4. *Intensity According to Wave Optics*

To calculate the intensity on and near the caustics we will use the Kirchoff integral. Although this integral assumes that the wave is scalar, it can be applied to the present (tensor) case because we assume that the angles θ_1 and θ_2 (see Fig. 2) are small. The planes of polarization associated with the different rays are then nearly parallel and the waves add nearly as scalars.†

† The polarization tensor undergoes parallel transport along the null geodesic. It is easy to check that this implies that there is no rotation of the polarization about the z-axis; there is a rotation by $\theta/2$ about the y-axis.

The Kirchoff integral (Born & Wolf, 1970) gives for the wave amplitude at the point P

$$A_p = -\frac{A_0 i}{\lambda}\frac{1}{l_2}\iint e^{i\delta(b)} e^{ikf}\, d\xi\, d\eta \tag{4.1}$$

where

$$k = 2\pi/\lambda \tag{4.2}$$

$$f = l_1 + l_2 - x\xi/l_2 - y\eta/l_1 + \tfrac{1}{2}(1/l_1 + 1/l_2)(\xi^2 + \eta^2) + \cdots \tag{4.3}$$

and where (x, y, l_2) are the coordinates of the point P, $(0, 0, -l_1)$ are the coordinates of the source S, and (ξ, η) are the coordinates of a point in the x–y plane; the center of the star is located at $\xi = \eta = 0$ (see Fig. 3). The

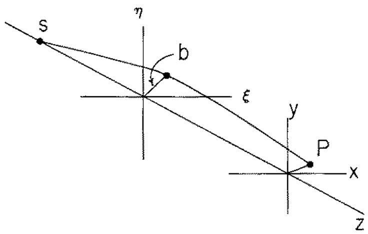

Figure 3.—Arbitrary path from source to observer.

quantity A_0 gives the amplitude of the wave reaching the star. Equation (4.1) differs from the usual Kirchoff integral in that an extra factor $\exp(i\delta(b))$ appears in the integrand; $\delta(b)$ is the phase shift that the gravitational field produces in the wave along a ray of impact parameter b; to a sufficient approximation we can take $b^2 = \xi^2 + \eta^2$. In equation (4.3) terms of order $1/l^2$ have been neglected.

The expression (4.1) for A_p is an adequate approximation only if the wavelength λ is small compared to the size of the 'lens'. If the wavelength is large, it is necessary to take into account the scattering of the gravitational wave by the curvature of spacetime.

Introducing polar coordinates in the planes $z = 0$ and $z = l_2$,

$$\xi = b\cos\phi \qquad \eta = b\sin\phi \tag{4.4}$$

$$x = l_2 w\cos\psi \qquad y = l_2 w\sin\psi \tag{4.5}$$

we can transform (4.1) into (omitting irrelevant phase factors)

$$A_p \propto \frac{A_0}{\lambda}\frac{2\pi}{l_2}\int \exp[i\delta(b)]\, J_0(kbw)\exp[\tfrac{1}{2}ik(1/l_1 + 1/l_2)b^2]\, b\, db \tag{4.6}$$

If k is sufficiently large, this integral can be approximately evaluated by the *method of stationary phase* (Born & Wolf, 1970) which only takes into account the contribution from the region near the geometrical ray (or rays) connecting S and P. Consider first the case $kb_0 w \ll 1$ (where b_0 is that value of b which makes the phase stationary). We can approximate

$$J_0(kb_0 w) \simeq 1 - (kb_0 w)^2/4 \tag{4.7}$$

and the condition of stationary phase is

$$\frac{d}{db}\left[\tfrac{1}{2}k(1/l_1 + 1/l_2)b^2 + \delta(b)\right] = 0 \tag{4.8}$$

By looking at the behavior of wave fronts, it is easy to see that the phase shift is related to the deflection angle by

$$\theta(b) = -\frac{1}{k}\frac{d\delta}{db} \tag{4.9}$$

Hence equation (4.8) becomes

$$b/l_1 + b/l_2 - \theta(b) = 0 \tag{4.10}$$

which is precisely the relation between b and θ required by geometrical optics for a ray that connects S and P (with $w = 0$). If we designate the solution of this equation by b_0, then the phase of the integrand in the vicinity of b_0 is given by

$$\begin{aligned}\delta(b) + \tfrac{1}{2}k(1/l_1 + 1/l_2)b^2 &\simeq \delta(b_0) + k(1/l_1 + 1/l_2)b_0^2 \\ &+ \tfrac{1}{2}k(b - b_0)^2[1/l_1 + 1/l_2 - \theta'(b_0)] + \cdots\end{aligned} \tag{4.11}$$

The contribution to the integral (4.6) from the region near $b = b_0$ is then (omitting again an irrelevant phase factor)

$$A_p \propto \frac{A_0}{\lambda}\frac{2\pi b_0}{l_2}\left\{\frac{\pi}{\tfrac{1}{2}ik[1/l_1 + 1/l_2 - \theta'(b_0)]}\right\}^{1/2}[1 - (kb_0 w)^2/4] \tag{4.12}$$

and the intensity gain

$$(\text{gain, near axis}) = \frac{4b_0^2\pi^2[1 - (kb_0 w)^2/4]^2}{\lambda|1/l_1 + 1/l_2 - \theta'(b_0)|}\left(\frac{l_1 + l_2}{l_1 l_2}\right)^2 \tag{4.13}$$

A necessary condition for the validity of the approximation is that $kb_0^2[1/l_1 + 1/l_2 - \theta'(b_0)] \gg 1$. The phase of the integrand is also stationary at $\theta = 0$, $b_0 = 0$. It is easy to check that the contribution to the integral form of this region is negligible compared to (4.12).

Equation (4.13) *roughly* indicates that the intensity gain has a large value only for an angular distance

$$w \simeq 2/kb_0 = \lambda/\pi b_0 \tag{4.14}$$

away from the axis. Obviously we are dealing with a typical diffraction peak.

In the case $kb_0 w \gg 1$, we can approximate

$$J_0(kbw) \simeq [e^{i(kbw - \pi/4)} + e^{-i(kbw + \pi/4)}]/\sqrt{(2\pi kbw)} \tag{4.15}$$

The phase is then stationary at

$$b/l_1 + b/l_2 - \theta(b) \pm w = 0 \tag{4.16}$$

The $\pm$ signs in (4.16) correspond, respectively, to rays that reach P after and before crossing the axis. In what follows we will ignore the interference fringes and only indicate the contribution due to one of the solutions of equation (4.16). The intensity gain is

$$\frac{4b^2\pi^2}{\lambda|1/l_1 + 1/l_2 - \theta'(b_0)|}\frac{1}{2\pi k b_0 w}\left(\frac{l_1+l_2}{l_1}\right)^2$$

$$= \frac{b_0}{hl_2}\frac{1}{|1/l_1 + 1/l_2 - \theta'(b_0)|}\left(\frac{l_1+l_2}{l_1}\right)^2 \tag{4.17}$$

in agreement with result (3.4) of geometrical optics.

If both

$$1/l_1 + 1/l_2 - \theta/b = 0 \tag{4.18}$$

and

$$1/l_1 + 1/l_2 - \theta' = 0 \tag{4.19}$$

then equation (4.13) fails. This corresponds to the point P lying on the intersection of the caustic line $w = 0$ with the caustic surface; this point is the *image.* The consistency of (4.18) and (4.19) demands

$$\theta' = \theta/b \tag{4.20}$$

Obviously this condition is always satisfied at $b = 0$ where $\theta(b)$ can be approximated by the expression (const.) x b. Since, as is easy to check with equation (2.5), $\theta''(O) = 0$, the relevant term in the power series expansion for the phase is the fourth-order term

$$\delta(b) + k(1/l_1 + 1/l_2)b^2 \simeq \delta(O) - kb^4\theta'''(O)/24 + \cdots \tag{4.21}$$

The integral (4.6) then gives, at $w = 0$,

$$(\text{gain, at image}) = \frac{3\pi^2}{\lambda}\left(\frac{l_1+l_2}{l_1 l_2}\right)^2 \frac{1}{|\theta'''(O)|} \tag{4.22}$$

In equations (4.20) and (4.21) there appear the derivatives θ' and θ''' evaluated at $b = 0$. Although these can be found once $\theta(b)$ is given, it is convenient to express these derivatives directly in terms of the mass density. Differentiation of (2.5) yields

$$\theta'(O) = \frac{8\pi G}{c^2}\int_0^R \rho\, dr \tag{4.23}$$

$$\theta'''(O) = \frac{12\pi G}{c^2}\int_0^R \frac{1}{r}\frac{d\rho}{dr}\, dr \tag{4.24}$$

Finally, we want the intensity on the caustic surface. Points on this surface simultaneously satisfy the conditions

$$1/l_1 + 1/l_2 - \theta/b \pm w/b = 0 \tag{4.25}$$

and

$$1/l_1 + 1/l_2 - \theta' = 0 \tag{4.26}$$

If we use the approximation (4.15) for $J_0(kbw)$, the phase of the integrand is

$$\delta(b) \pm kbw \mp \pi/4 + k(1/l_1 + 1/l_2)b^2 \simeq (\text{const.}) - k(b - b_0)^3 \theta''(b_0)/6 + \cdots \tag{4.27}$$

where b_0 is the solution of (4.25) and (4.26). This results in an intensity gain

$$(\text{gain, on caustic}) = \frac{1}{\lambda^{1/3}} \left(\frac{l_1 + l_2}{l_1 l_2}\right)^2 \frac{b_0}{w} \frac{[\Gamma(1/3)]^2}{|9\pi\theta''(b_0)|^{2/3}} \tag{4.28}$$

The intensity in the vicinity of the image can be calculated by using the approximation (4.7); this leads to the following rough estimate of the width of the peak,

$$w \simeq \frac{\lambda}{\pi} \left| \frac{\pi^2 \theta'''(o)}{12\lambda} \right|^{1/4} \tag{4.29}$$

5. *Some Examples*

We will present some numerical estimates of the focusing produced by objects of large mass. The mass distributions we will investigate are (i) a polytrope of index $n = 3$, and (ii) a sphere of uniform density. In the following calculations we assume that $l_2 \gg l_1$ since this is the relevant case for an observer on the earth. Assumptions inherent in the calculations of Section 4 are a small deflection angle ($b/l_1 \ll 1$) and the stationary phase approximation ($kb^2\theta' \gg 1$ for the case of equation (4.13)).

(i) *Polytrope, $n = 3$*

In an equilibrium star with mass larger than $10^2 M_\odot$, the pressure and internal energy are determined mainly by radiation (Zel'dovich & Novikov, 1965). This implies that to a first approximation such a star can be described as a polytropic gas sphere with $n = 3$ and density given by equation (2.6) (for this equilibrium configuration the mass and the radius are *independent parameters*).

For rays of sufficiently large impact parameter, the deflection angle is

$$\theta = 4GM/bc^2$$

This expression is, of course, exact for $b \geqslant R$, but it also gives a good approximation for $b \geqslant 0{\cdot}5R$ (see Fig. 1). This is not surprising since, in our example, the mass is sharply concentrated near the center. With $\theta = b/l_1$ we find that

$$b = \sqrt{(4GMl_1/c^2)}; \qquad \theta = \sqrt{(4GM/l_1c^2)} \tag{5.1}$$

and hence the gain is

$$8\pi^2 GM/\lambda c^2 \tag{5.2}$$

This depends only on the mass of the star and on the wavelength of the gravitational radiation; it does not depend on l_1 or l_2. Taking the value $\lambda = 1{\cdot}8 \times 10^7$ cm (corresponding to the frequency $\nu = 1{\cdot}66 \times 10^3$ sec^{-1} to which Weber's apparatus is tuned), results in a gain of†

$$0{\cdot}65M/M_\odot \tag{5.3}$$

The width of the diffraction peak is roughly $\lambda/\pi b$. By (5.1), this is

$$\frac{\lambda}{\pi}\sqrt{\left(\frac{c^2}{4GMl_1}\right)} \sim 2{\cdot}8 \times 10^{-2}\sqrt{\left(\frac{R_\odot}{l_1}\frac{M_\odot}{M}\right)} \tag{5.4}$$

where $R_\odot \simeq 6{\cdot}96 \times 10^{10}$ cm. If we assume that $l_1 \simeq 100R_\odot$ (a *much* smaller value is unlikely; in any case (5.4) is not very sensitive to a change in l_1 by a factor of ten or so), then the width is

$$2{\cdot}8 \times 10^{-3}\sqrt{(M_\odot/M)} \tag{5.5}$$

Obviously, the width is small whenever the gain is large. Although with an increase of mass the width of the central diffraction peak shrinks, the gain at a given angle w *increases with mass* (we ignore interference fringes).

The preceding does not depend explicitly on the value of R. However, R must satisfy the condition

$$R < 0{\cdot}06R_\odot\sqrt{(M/M_\odot)} \tag{5.6}$$

The reason is that according to Fig. 1 rays with $b > 0{\cdot}5R$ have $\theta/b < 15GM/R^2c^2$. This leads to (5.6) if we combine it with $\theta/b \simeq 1/l_1 = 1/100R_\odot$.

We remark that (5.3) and (5.5) do not really depend on the mass distribution. These results are valid whenever the rays pass outside of (or outside most of) the mass. For example, our 'lens' might consist of the gravitational field surrounding a black hole.

Rays that pass near the center of the star contribute very little to the intensity *unless* the distances l_1 and l_2 are related as in equation (4.19). This means the observer is at the image of the source. Since l_2 is taken as very large (image at infinity), only the value of l_1 is critical; equation (4.19) reduces to

$$l_1 = 1/\theta' \tag{5.7}$$

We may say that the source is at the *focus*, the focal distance being $1/\theta'$. For our example, equations (4.23) and (4.24) yield

$$\theta'(O) = 58GM/R^2c^2 \tag{5.8}$$

$$\theta'''(O) = -4{\cdot}1 \times 10^3 GM/R^4c^2 \tag{5.9}$$

† The condition $|kb^2\theta'| \gg 1$ becomes $8\pi GM/\lambda c^2 \gg 1$ and is therefore satisfied whenever the gain is large. Thus, (5.3) is valid only if $M/M_\odot \gg 1$.

and hence the intensity gain (4.22) becomes

$$0{\cdot}20M/M_\odot \tag{5.10}$$

This again depends only on the mass of the star. The result (5.10) is smaller, and therefore less interesting, than (5.3). A strong image is possible only if the mass distribution is such that $\theta(b)$ can be very well approximated by a linear function over a fairly large region around the origin; this region then behaves as a well-corrected lens. For a mass distribution that is concentrated near the origin (such as our polytrope of index $n = 3$) these conditions are not satisfied. For a uniform mass density, the conditions are somewhat more favorable and the gain due to nearly central rays is somewhat larger than (5.3) (see below).

For our mass distribution, the intensity gain (4.28) on the caustic surface is *not* very large. Even for the case $M/M_\odot = 10^5$, the gain calculated from equations (4.28) is only a factor of three.

(ii) *Uniform Sphere*

Although spherical mass distributions of uniform density are of little astrophysical interest, the focusing produced by such a mass distribution may be of some interest because it is the same as that produced by the thin rotating massive discs of Bardeen & Wagoner (1971). In the non-relativistic case, the rotating thin disc has a mass distribution which is exactly that obtained by projecting the mass of a uniform sphere on the plane of the disc. Hence, for rays incident parallel to the axis of symmetry, the deflections, and gains, produced by sphere and disc are the same.

For a sphere of uniform density, the intensity gain is largest if the source is at the focus (equation (5.7)). The values of $\theta'(O)$ and $\theta'''(O)$ are

$$\theta'(O) = 6GM/R^2c^2 \tag{5.11}$$

$$\theta'''(O) = -9GM/R^4c^2 \tag{5.12}$$

which results in a gain of

$$0{\cdot}98M/M_\odot \tag{5.13}$$

The width of the region of high intensity is given by equation (4.29). To calculate this width we must first specify the value of R. We will arbitrarily fix R by the condition

$$GM/Rc^2 = 10^{-2} \tag{5.14}$$

This condition represents a compromise: we want a small value of R in order to obtain a large width, but we must keep GM/Rc^2 reasonably small since otherwise our linear approximation fails. The width is then

$$1{\cdot}1(M_\odot/M)^{3/4} \tag{5.15}$$

According to (5.7) and (5.11), the value of l_1 must be

$$l_1/R = Rc^2/6GM \simeq 17 \tag{5.16}$$

so that the focus is quite near the star.

Stronger focusing can be achieved by correcting the aberration of the 'lens'. One way to do this is by increasing the mass density of the outer layers. Figure 1 shows the deflection angle for a thin shell of radius r ($\rho(r) = M\delta(r - R)/4\pi R^2$). Obviously a superposition of shell and uniform sphere will give improved focusing since the corresponding $\theta(b)$ is linear out to larger values of b. Combining a sphere of mass M, and a shell of mass $2{\cdot}7M$ increases the gain (5.13) by a factor of ten at the expense of reducing the width by a factor of two. The *relativistic* thin rotating discs of Bardeen and Wagoner have more mass along the outer edge than the non-relativistic discs; however, extrapolation of our results to the relativistic case is questionable.

6. *Conclusion*

Our examples show that the focusing of gravitational radiation can easily give very large gains in the intensity, but that these large gains only occur in a beam of small angular width. This means that if a focusing mechanism was involved in the production of a burst of gravitational radiation at the galactic center, then the amount of energy emitted in this burst was much lower than that calculated on the assumption of isotropy of the intensity received at the earth. However, the total amount of energy emitted by the source in a period of, say, one year would still be comparable to, or *larger* than, that calculated on the assumption of isotropy, since it is to be expected that many bursts of radiation miss the earth because the source is not correctly aligned.

One way to reduce the energy requirements on our galaxy is to suppose that the gravitational radiation is extragalactic. For example, if we suppose that some other galaxy contains a thin rotating disc of mass $\sim 10^9 M_\odot$ in its core and a source at the focal point (size of focal 'point' is $\sim 10^{11}$ cm wide, 10^{15} cm long†), then the focused intensity at the earth will be the same as for an isotropic source of equal energy at the center of our galaxy if the distance to the other galaxy is 6×10^8 light years. However, the width of the high intensity beam is only 2×10^{-7} radians and the *a priori* probability of the earth sitting in this beam is negligible (Ohanian, 1973).

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† The longitudinal size of the focal 'point' is approximately $\Delta l_1 \sim w^2 l_1^2/\lambda$ (Sommerfeld, 1964) where w is the width given in equation (5.15).

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